Localization of classical waves in a random medium: A self-consistent theory

J. Kroha
Institut für Theorie der Kondensierten Materie, Universität Karlsruhe, 7500 Karlsruhe 1, Germany

C. M. Soukoulis
Ames Laboratory and Department of Physics, Iowa State University, Ames, Iowa, 50011

P. Wölfle
Institut für Theorie der Kondensierten Materie, Universität Karlsruhe, 7500 Karlsruhe 1, Germany
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We study localization of classical waves in a model of point scatterers, idealizing a random arrangement of dielectric spheres ($\epsilon=1+\Delta\epsilon$) of volume V_s and mean spacing a in a matrix ($\epsilon=1$). At distances $\gg a$ energy transport is diffusive. A self-consistent equation for the frequency-dependent diffusion coefficient is obtained and evaluated in the approximation where noncritical quantities are calculated in the coherent potential approximation. The velocity of energy transport and the phase velocity are renormalized in a similar way, even for finite-size scatterers. We find localization for $d=3$ dimensions in a frequency window centered at $\omega\simeq 2\pi/a$, and for values of the average change in the dielectric constant $\overline{\Delta\epsilon}=(V_s a^{-3})\Delta\epsilon$ exceeding ~ 1.7.

In recent years there has been growing interest in studies of the propagation of classical waves in random media.[1,2] The revival of interest in the long-standing problem of multiple scattering of classical waves was initiated by the discovery of the importance of quantum interference effects for the transport properties of electrons in disordered systems.[3] While some of the features associated with electron localization, such as enhanced coherent backscattering, have been detected in light scattering experiments[4] as well, the localization of electromagnetic waves or other classical waves in random systems has not been established beyond doubt. The question of localization of classical waves has attracted attention for two reasons. First, the properties of classical waves such as light waves, microwaves, and acoustic waves in random media are of fundamental interest for their own sake. Second, classical waves can serve as a model system for testing the theory of Anderson localization of electrons experimentally in a clean way, without the complication of strong inelastic scattering and other effects of electron-electron and electron-phonon interaction. On the other hand, it is harder to localize classical waves, mainly due to the fact that at low frequency the effect of disorder tends to average out in this case, whereas electrons at low energy are trapped more effectively, even by a weak random potential. Existing theories predict localization of classical waves under certain circumstances.[5–8] However, there is no conclusive experimental evidence yet, although the recent experiments by Genack and collaborators provide strong indications for the existence of localization of light.[9]

The outstanding problem in classical wave localization is to find the optimal conditions for its realization. It has been suggested that an intermediate-frequency window of localized states separates the low-frequency extended states characterized by Rayleigh scattering from the high-frequency extended states described by geometric optics. Theories based on the weak scattering limit and on the coherent potential approximation (CPA) predict frequency intervals within which localization should be observed.[7,8] These predictions are based on extrapolation of results, obtained in the weak disorder regime. In addition it was recently recognized that considerable care has to be exercised in transforming the results of the theory of localization of electrons to the case of classical waves.[10,11]

This somewhat difficult situation has led to suggestions of alternative pathways to localization. John[12] has proposed that classical localization may be more easily achieved for a weakly disordered system of periodically arranged dielectric structures in the frequency regime near a band gap. The question of photonic band structure in periodic dielectric structures is a fascinating subject with potential applications in the telecommunication, information processing, optical storage, and sensor technology.

In this paper we present a self-consistent theory of localization of classical waves, similar to the one developed for electron localization.[13–15] There are important differences in the formulation of such a theory for classical waves as compared to the case of electrons. The most important one is that the equivalent of particle density is not conserved. Unfortunately, this fact has not been given sufficient attention in an earlier attempt[7] to derive a self-consistent theory, as pointed out recently.[10] The quantity conserved here is energy, not mass, leading to diffusion behavior of the energy density. Another difference is that the scattering potential is energy dependent. As a consequence, the energy transport velocity entering the diffusion coefficient for a strongly disordered system may be appreciably renormalized,[10] and, consequently, diffusion coefficients can be quite small even far

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from the localization transition. Also, in previous versions of a self-consistent theory of localization[7] the single-particle quantities and the coupling constants were calculated in the low density or the weak scattering approximation. It is known from the electron transport problem[14] how to improve upon this unnecessary simplification by calculating the noncritical quantities in CPA.

For simplicity, we will in the following consider an idealization of a real system, such as dense packed dielectric spheres of random size scattering classical waves, for example, light: a system of point scatterers distributed statistically throughout the system. Then the only length scale characterizing the system is the mean spacing of neighboring scatters a. In the two limits of wavelength λ of the incident wave small or large in comparison with a, one expects effective-medium theory to work well, whereas for $\lambda=2\pi/\omega\simeq a$ localization of waves should occur for sufficiently large scattering strength.

We consider the propagation of classical waves in a random medium described by the wave equation for the scalar field $\Psi(r,t)$:

$$\left[\epsilon(r)\frac{\partial^2}{\partial t^2}-\nabla^2\right]\Psi(r,t)=0 \ . \tag{1}$$

Here $\epsilon(\mathbf{r})$ characterizes the randomly varying phase velocity $c(\mathbf{r})=1/\sqrt{\epsilon(\mathbf{r})}$. In the approximation where the vector nature of the electromagnetic field is neglected, (1) describes the propagation of electromagnetic waves in a dielectric medium with spatially varying dielectric constant $\epsilon(\mathbf{r})$. We will assume the fluctuations of $\epsilon(\mathbf{r})$ to be spatially uncorrelated, such that $\langle\epsilon(\mathbf{r})\rangle=\bar{\epsilon}$ and $\langle\epsilon_1(\mathbf{r})\epsilon_1(\mathbf{r}')\rangle=W\delta(\mathbf{r}-\mathbf{r}')$, where $\epsilon_1(\mathbf{r})=\epsilon(\mathbf{r})-\bar{\epsilon}$.

The (unaveraged) Green's function of the wave equation (1), Fourier-transformed with respect to time, satisfies the Dyson equation

$$G(\mathbf{r},\mathbf{r}';\omega)=G^0(\mathbf{r},\mathbf{r}';\omega)+\int d^dr''G^0(\mathbf{r}-\mathbf{r}'';\omega)U(\mathbf{r}'';\omega)G(\mathbf{r}'',\mathbf{r}';\omega) \tag{2}$$

with the "interaction potential" $U(\mathbf{r},\omega)=-\omega^2[\epsilon(\mathbf{r})-1]$ and the free propagator G^0 given by its Fourier transform $G_{\mathbf{k}}^{0R,A}(\omega)=[(\omega\pm i0)^2-k^2]^{-1}$.

In the model of point scatterers $\epsilon(\mathbf{r})$ is given by $\epsilon(\mathbf{r})-1=(\Delta\epsilon V_s)\sum_{i=1}^{N_I}\delta(\mathbf{r}-\mathbf{r}_i)$, with N_I the number and $n_I=N_I/V=1/a^d$ the density of scatters (d is the spatial dimension). The point scatterers may be thought of as an idealization of spheres of dielectric material of volume V_s and dielectric constant $1+\Delta\epsilon$ embedded in a matrix with $\epsilon=1$. The two parameters characterizing the model may be chosen as the average dielectric constant $\bar{\epsilon}=1+(V_s/a^d)\Delta\epsilon$ and the average separation of scattering centers a.

The impurity-averaged Green's function $G_{\mathbf{k}}^{R,A}(\omega)$ is given in terms of the mass operator $\Sigma_{\mathbf{k}}^{R,A}(\omega)$ by $G_{\mathbf{k}}^{-1}=G_{\mathbf{k}}^{0-1}(\omega)-\Sigma_{\mathbf{k}}$. The one-particle properties are expected to be uncritical at the localization transition and hence may be calculated in CPA, i.e., in single-site approximation[16,4] when $\Sigma_{\mathbf{k}}(\omega)=\Sigma_0(\omega)$ is independent of the wave vector.

The transport properties of the system can be extracted from the averaged two-particle Green's function

$$\Phi_{\mathbf{kk}'}^{\omega}(\mathbf{q},\Omega)=-\langle G^R(\mathbf{k}_+,\mathbf{k}'_+;\omega_+)G^A(\mathbf{k}'_-,\mathbf{k}_-;\omega_-)\rangle \ ,$$

where $\omega_\pm=\omega\pm\Omega/2$, $\mathbf{k}_\pm=\mathbf{k}\pm\mathbf{q}/2$, $\mathbf{k}'_\pm=\mathbf{k}'\pm\mathbf{q}/2$, and $\Omega,\mathbf{q}$ are the center-of-mass frequency and wave vector, respectively. The long-wavelength–low-frequency behavior of the classical wave system is, in contrast to the electronic case, not governed by particle number conservation, but by the conservation of energy, with $\epsilon(\partial\Psi/\partial t)^2$ being the energy density of the wave field. The quantity $\Phi_{\epsilon\epsilon}^{\omega}=(\omega/c_{\rm ph})^2\sum_{\mathbf{kk}'}\Phi_{\mathbf{kk}'}^{\omega}(\mathbf{q},\Omega)$ may be interpreted as an energy density correlation function, where the phase velocity $c_{\rm ph}$ is defined from the zero of the real part of the inverse of $G_{\mathbf{k}}(\omega)$ as $c_{\rm ph}^{-2}(\omega)=1-{\rm Re}\Sigma_0(\omega)/\omega^2$. $\Phi_{\epsilon\epsilon}^{\omega}$ can be shown to have the diffusion pole structure

$$\Phi_{\epsilon\epsilon}^{\omega}(\mathbf{q},\Omega)=c(\omega)\frac{i(\omega/c_{\rm ph}){\rm Im}G_0^A(\omega)}{\Omega+iD(\Omega)q^2} \tag{3}$$

in the limit Ω, $q\to 0$ [$c(\omega)$ and $G_0^A(\omega)$ will be defined below]. In the regime of localized waves, the diffusion coefficient $D(0)$ vanishes identically. In the following we swill calculate $D(\Omega)$ as a function of disorder and show that a localization transition takes place in the model of point scatterers in the regime $\omega\simeq 2\pi/a$ for sufficiently strong coupling $\bar{\epsilon}$.

The starting point for a calculation of the averaged two-particle Green's function $\Phi_{\mathbf{kk}'}^{\omega}(\mathbf{q},\Omega)$ is the Bethe-Salpeter equation

$$\Phi_{\mathbf{kk}'}^{\omega}(q,\Omega)=G_{\mathbf{k}_+}^R(\omega_+)G_{\mathbf{k}_-}^A(\omega_-)\times\left[\delta_{\mathbf{k},\mathbf{k}'}+\sum_{\mathbf{k}''}\gamma_{\mathbf{kk}''}^{\omega}(\mathbf{q},\Omega)\Phi_{\mathbf{k}''\mathbf{k}'}^{\omega}(q,\Omega)\right] \ , \tag{4}$$

where $\gamma_{\mathbf{kk}''}^{\omega}(q,\Omega)$ is the sum of all irreducible diagrams of the four-point vertex function. We can write (4) as a kinetic equation with the help of the Ward identity

$$\Sigma_{\mathbf{k}_+}^R-\Sigma_{\mathbf{k}_-}^A=\sum_{\mathbf{k}'}\gamma_{\mathbf{kk}'}^{\omega}[G_{k'_+}^R-G_{k'_-}^A]+\frac{\omega\Omega}{\omega^2+(\Omega/2)^2}\left[\Sigma_{\mathbf{k}_+}^R+\Sigma_{\mathbf{k}_-}^A+\sum_{\mathbf{k}'}\gamma_{\mathbf{kk}'}^{\omega}(G_{k'_+}^R+G_{k'_-}^A)\right] \ , \tag{5}$$

where $\Sigma_{k_+}^R=\Sigma_{\mathbf{k}_+}^R(\omega_+)$, etc. Compared to the case of electrons, there is an additional (the last) term on the rhs of (5), which has been missed in Ref. 7. This term arises because of the explicit frequency dependence of the perturbation $U(\mathbf{r},\omega)\propto\omega^2$ in (2).

Energy conservation is expressed by the equation

$$\Omega\Phi_{\epsilon\epsilon}^{\omega}-q\Phi_{j\epsilon}^{\omega}=ic(\omega)\frac{\omega}{c_{\rm ph}}{\rm Im}G_0^A \ , \tag{6}$$

where the energy current correlation function corre-

sponding to $\Phi^{\omega}_{\epsilon\epsilon}$ has been defined as

$$\Phi^{\omega}_{j\epsilon}=c(\omega)\frac{\omega}{c_{\rm ph}}\sum_{\mathbf{k},\mathbf{k}'}(\mathbf{k}\cdot\hat{\mathbf{q}})\Phi^{\omega}_{\mathbf{k}\mathbf{k}'}(q,\Omega)\ .$$

Here we have introduced a renormalized velocity $c(\omega)$ characterizing energy diffusion. In the limit of small q, Ω, $c(\omega)$ is given[17] by

$$\left[\frac{c(\omega)}{c_0}\right]^{-1}=\left[\frac{c_{\rm ph}}{c_0}\right]\left[1-\frac{1}{\omega^2}(\gamma_0{\rm Re}G_0^A+{\rm Re}\Sigma_0^A)\right]\ , \quad (7)$$

where $G_0^A(\omega)=\sum_{\mathbf{k}}G_{\mathbf{k}}^A(\omega)$, $\gamma_0={\rm Im}\Sigma_0^A/{\rm Im}G_0^A$ is the irreducible vertex function calculated in CPA, and $c_0\equiv 1$ is the phase velocity in the homogeneous background medium. The renormalization of c is a consequence of the additional term in the Ward identity (5), which is of order Ω, while energy conservation is already guaranteed by the $\Omega\rightarrow 0$ limit of (5). We have calculated $G_0(\omega)$ and $\Sigma_0(\omega)$ in CPA and evaluated $c(\omega)$ in this approximation. The result is shown in Fig. 1. At low frequencies $c(\omega)$ is substantially smaller than one (for positive $\Delta\epsilon$), as already denoted in Ref. 10, the limiting value as $\omega\rightarrow 0$ being given by $c(\omega)\rightarrow 1/\sqrt{\bar{\epsilon}}$, whereas at high frequency $c(\omega)\rightarrow 1$. The velocity of energy transport $c(\omega)$ and the phase velocity $c_{\rm ph}(\omega)$ are seen to agree well. In fact, whenever the imaginary part of the mass operator Σ is small (e.g., in the limit of low density of scatterers), c and $c_{\rm ph}$ coincide. This is seen immediately from (7) for the case of point scatterers, and was shown in Ref. 11 for general momentum-dependent scattering within a low-density approximation. Therefore, one may conclude that the strong reduction of $c(\omega)$ with respect to $c_{\rm ph}$, as measured experimentally by van Albada *et al.*,[10] is purely an effect of resonant scattering: In the vicinity of resonances ${\rm Im}\Sigma$ is always large even for small density and may cause the reduction of $c(\omega)$.

Let us turn to localization effects now. Employing the techniques developed for the case of electron transport,[14,15] one may derive an equation of motion for $\Phi_{j\epsilon}$ from the Bethe-Salpeter equation (4), which relates back to $\Phi_{\epsilon\epsilon}$ and hence allows one to derive the diffusion form (3) for $\Phi_{\epsilon\epsilon}$. In the approximation for the irreducible vertex $\gamma_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$ employed in the case of electron transport,[14,15]

$$\gamma_{\mathbf{k}\mathbf{k}'}(\mathbf{q},\Omega)=\frac{1}{\omega^2}({\rm Im}\Sigma_0^A)^2({\rm Im}G_0^A)^{-3}{\rm Im}G_{\mathbf{k}}^A(\omega)\Phi^{\omega}_{\epsilon\epsilon}(|\mathbf{k}+\mathbf{k}'|,\Omega)\ ,$$

the diffusion coefficient $D(\Omega)$ is found to satisfy the self-consistency equation

$$D(\Omega)=\left[1-\frac{i\omega\Omega}{{\rm Im}\Sigma_0^A}\right]^{-1}\times\left[D_0+2c^2(\omega)\left[\frac{c_{\rm ph}}{\omega}\right]^2\frac{{\rm Im}\Sigma_0^A}{({\rm Im}G_0^A)^2D_0}\times K\left[\frac{-i\Omega}{D(\Omega)}\right]\right]\ , \quad (8)$$

where

$$K\{z\}=\sum_{\mathbf{k},\mathbf{k}'}(\mathbf{k}\cdot\hat{\mathbf{q}})\frac{{\rm Im}G_{\mathbf{k}}^A({\rm Im}G_{\mathbf{k}'}^A)^2}{z+(\mathbf{k}+\mathbf{k}')^2}(\mathbf{k}'\cdot\hat{\mathbf{q}})\ . \quad (9)$$

In CPA the bare diffusion constant is given by

$$D_0=2c(\omega)\frac{c_{\rm ph}}{\omega}\frac{1}{{\rm Im}G_0^A}\sum_{\mathbf{k}}(\mathbf{k}\cdot\hat{\mathbf{q}})^2({\rm Im}G_{\mathbf{k}}^A)^2\ .$$

We have solved (8) numerically for $D(\Omega)$, using the CPA results for G_0, Σ_0, $c(\omega)$, and D_0. It is easily shown that in CPA the parameter a can be absorbed into the rescaled frequency $a\omega$, so that the only relevant parameter in the model is the average dielectric constant $\bar{\epsilon}$.

In Fig. 2, we present the mobility edge trajectory separating extended from localized states. Notice that the model correctly describes the underlying physics. In particular, there are only extended states for all frequencies ω when $\bar{\epsilon}$ is less than 2.7. In addition, for low (Rayleigh

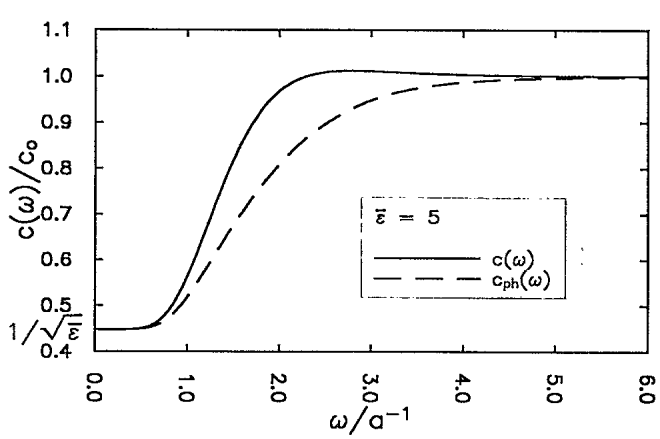

FIG. 1. The transport velocity $c(\omega)$, normalized to c_0, the phase velocity in the homogeneous background medium, is shown for $\bar{\epsilon}=5$. Also shown is the effective phase velocity $c_{\rm ph}(\omega)$ in the disordered system. $c(\omega)$ and $c_{\rm ph}(\omega)$ coincide for $\omega\ll 2\pi/a$ and for $\omega\gg 2\pi/a$ or in the limit $\Delta\epsilon\rightarrow 0$ for all frequencies.

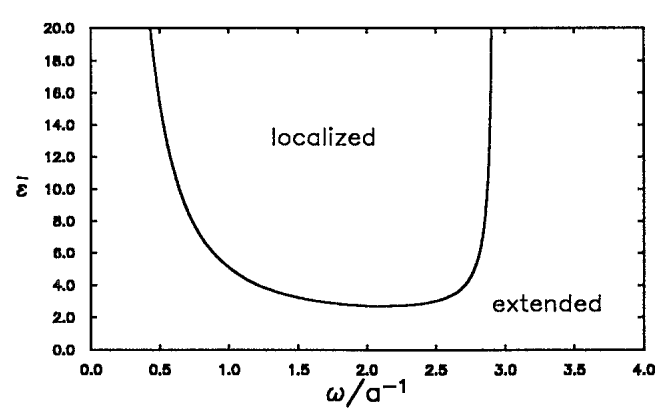

FIG. 2. The mobility edge trajectory in the dielectric constant frequency plane separating extended from localized states is shown.

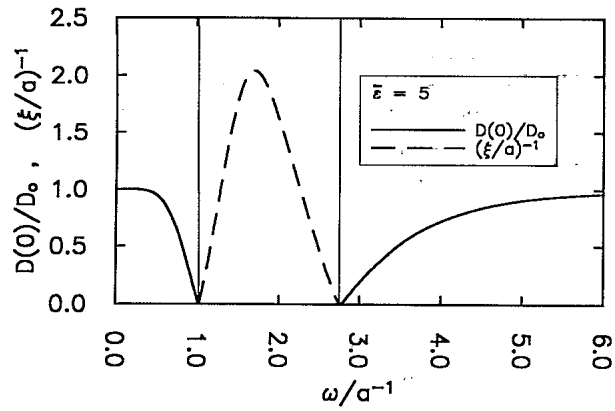

FIG. 3. The diffusion coefficient $D(0)$ and the inverse localization length ξ^{-1} are shown as a function of wave frequency ω for $\bar{\epsilon}=5$.

scattering) and high (geometric optics) frequencies, as expected, there are only extended states for any value of $\bar{\epsilon}$.

In Fig. 3, the dc diffusion coefficient $D(0)$ for spatial dimension $d=3$ is shown as a function of wave frequency ω for $\bar{\epsilon}=5$. For values $\bar{\epsilon}\gtrsim 2.7$ [corresponding to energy velocity $c(0)\lesssim 0.6$], $D(0)$ is found to be zero within a frequency window centered at $a\omega\simeq 2$. One may define the localization length ξ by $\xi^2=\lim_{\Omega\to 0}D(\Omega)/(-i\Omega)$, which characterizes the spatial extension of wave packets localized in the system. $(\xi/a)^{-1}$ is also shown in Fig. 3 as a function of ω. $D(0)$ is seen to vanish linearly as ω approaches the critical frequencies $\omega_{1,2}$, whereas $\xi\propto|\omega-\omega_{1,2}|^{-1}$. Note that from (8) $\omega_{1,2}$ is independent of the renormalization of the transport velocity $c(\omega)$. The critical exponents for $D(0)$ and ξ found here, $s=1$ and $\nu=1$, are the same as those for the electron case. In dimensions $d\leq 2$ classical waves are found to be localized for arbitrarily weak disorder.

In the region where the wavelength is large compared to the size of the scatters, i.e., where the approximation by point scatters is appropriate, the description of the localization transition as a function of $a\omega$ and $\bar{\epsilon}$ presented here is expected to be semiquantitative, with the possible exception of a narrow critical regime, judging from the very good agreement of a similar theory for electron localization[13,14] with exact numerical results for finite systems. The predictions of the theory may be tested experimentally in systems with sufficiently large relative difference of dielectric constants $\Delta\epsilon$, such that the disorder parameter $\bar{\epsilon}$ may be large even for small density.

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[17]Equation (7) can be written in its more general form as

$$\frac{c_0}{c(\omega)}=\frac{c_{\rm ph}}{c_0}\left[1-\frac{1}{\omega^2}\left\{\sum_{k'}\gamma_{k_0k'}(q,\Omega){\rm Re}G_{k'}(\omega)+{\rm Re}\Sigma_{k_0}(\omega)\right\}\right],$$

where $k_0=\omega/c_{\rm ph}$, or by using the small $\mathbf{q}$ and Ω dependence of the Ward identity [Eq. (5)], the equation of the transport velocity is given by

$$\frac{c_0}{c(\omega)}=\frac{c_{\rm ph}}{c_0}\left[1-\frac{1}{2\omega}\frac{\partial}{\partial\omega}{\rm Re}\Sigma_{k_0}(\omega)-\frac{i}{\omega}\sum_{k'}\frac{\partial}{\partial\Omega}\gamma_{k_0k'}(0,\Omega){\rm Im}G'_k(\omega)+\frac{1}{2\omega}\sum_{k'}\gamma_{k_0k'}(0,0)\frac{\partial}{\partial\omega}{\rm Re}G_{k'}(\omega)\right],$$

which agrees with the expression given by Barabanekov and Ozrin (Ref. 11), but disagrees with Ref. 10, where the last term is missing.